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Transformation to pseudo-Cartesian coordinates in locally flat pseudo-Riemannian spaces

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Abstract

A tractable method is presented for obtaining transformations to pseudo-Cartesian coordinates in locally flat pseudo-Riemannian spaces. The procedure is based on the properties of parallel covector fields. As an illustration, the method is applied to obtain certain transformations that arise in the Hamilton–Jacobi theory of separation of variables.

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1. Introduction

This paper describes an apparently new method for deriving transformations to pseudo-Cartesian coordinates from general curvilinear coordinates in locally flat n-dimensional pseudo-Riemannian spaces. The method is based on the theory of parallel covector fields developed by Eisenhart [3], in particular on the result which states that a space admits n pointwise linearly independent parallel covector fields if and only if the Riemann curvature tensor vanishes. This result together with some elementary properties of such covector fields provides the basis for an algorithm that furnishes a straightforward procedure for the determination of transformations to pseudo-Cartesian coordinates.

Our method, which involves solving systems of linear differential equations, should be contrasted with the apparently more straightforward approach based on the tensor transformation equations for the metric tensor. One readily observes that this technique requires the solution of non-linear differential equations for the coordinate transformation which are generally intractable.

The motivation for this work comes from Eisenhart's moving frame method for determining all the orthogonally separable coordinate webs for the Hamilton–Jacobi (HJ) equation for the geodesics and the Schrödinger equation

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in Euclidean and pseudo-Euclidean spaces by integration of the equations for a valence-two Killing tensor [1,2,4,7–9]. The integration procedure yields expressions for the metric tensor and the Killing tensor in canonical separable coordinates. Since the Hamiltonian of a physical system is given in position–momentum coordinates, an essential step in the procedure for solving the equations of motion by the method of separation of variables is to find a coordinate transformation from canonical pseudo-Cartesian coordinates to canonical separable coordinates. The components of the general solution for the Killing tensor can then be obtained in terms of pseudo-Cartesian coordinates. The question of whether the HJ equation is indeed solvable by the method of separation of variables can consequently be answered by using the Invariant Theory of Killing Tensors which also facilitates the determination of the transformation to separable coordinates [6,10,11]. For $\mathbb{E}^2$, $\mathbb{E}^{1,1}$ and $\mathbb{E}^3$, this transformation may be found by inspection by exploiting the geometric properties of the orthogonally separable webs in each case. However, in the case of three-dimensional Minkowski space $\mathbb{E}^{2,1}$ [5], where there are more than fifty different cases to consider, this method breaks down. Thus we were led to search for a more systematic deductive method to find the required coordinate transformations.

The plan of the paper is as follows. In Section 2, we develop the theoretical basis and present our algorithm for deducing the transformation to pseudo-Cartesian coordinates. In Section 3, we apply our procedure to a simple example on the Euclidean plane and a less trivial example in three-dimensional Minkowski space.

2. Parallel covector fields on pseudo-Riemannian manifolds

Let $(\mathcal{M}, \boldsymbol{g})$ denote an n-dimensional real pseudo-Riemannian manifold. A covector field $\boldsymbol{\omega}$ is said to be *parallel* or *covariantly constant* (CC) if

$$\nabla_j \omega_i = 0, \tag{2.1}$$

where ω_i are the components of $\boldsymbol{\omega}$ with respect to a system of local coordinates on $\mathcal{M}$. Indeed, if ω_i^1 and ω_i^2 are CC, then

$$\omega_i^3 = a\omega_i^1 + b\omega_i^2 \tag{2.2}$$

is also CC, for any $a, b \in \mathbb{R}$. Moreover, if the set $\{\omega_i^1, \omega_i^2\}$ is pointwise linearly independent, $f, g \in C^1(\mathcal{M})$ and

$$\omega_i^3 = f\omega_i^1 + g\omega_i^2 \tag{2.3}$$

is CC, then the functions f and g are necessarily constant. It follows from (2.2) that the set of covariantly constant covector fields on $\mathcal{M}$ forms a vector space which we shall denote by $\mathcal{C}(\mathcal{M})$. Clearly we must have

$$\dim \mathcal{C}(\mathcal{M}) \leqslant n. \tag{2.4}$$

Proposition 1. $\dim \mathcal{C}(\mathcal{M}) = n$ *if and only if*

$$R^{\ell}{}_{ijk} = 0, \tag{2.5}$$

where $R^{\ell}{}_{ijk}$ *is the Riemann curvature tensor of the Levi-Civita connection on* $(\mathcal{M}, \boldsymbol{g})$.

Proof. ($\Longrightarrow$) Let $\{\omega_i^\alpha\}_{\alpha=1}^n$ be a basis of $\mathcal{C}(\mathcal{M})$. Any $\omega_i \in \mathcal{C}(\mathcal{M})$ may be expressed as $\omega_i = c_\alpha \omega_i^\alpha$, where c_α, $\alpha = 1, \ldots, n$, are arbitrary constants. By the Ricci identity applied to the basis covectors ω_i^α, we have

$$\omega_\ell^\alpha R^{\ell}{}_{ijk} = 0, \tag{2.6}$$

for $\alpha = 1, \ldots, n$. Let $x_0 \in \mathcal{M}$ be an arbitrary point. We may choose local coordinates such that

$$\mathring{\omega}_\ell^\alpha = \delta_\ell^\alpha. \tag{2.7}$$

It follows from (2.6) and (2.7) that $\mathring{R}^{\alpha}{}_{ijk} = 0$, for $\alpha = 1, \ldots, n$, i.e. the curvature tensor vanishes at x_0 in the special coordinate system for which (2.7) holds. Moreover, $\mathring{R}^{\ell}{}_{ijk} = 0$ in any system of coordinates containing x_0. We conclude that $R^{\ell}{}_{ijk} = 0$ at all points of $\mathcal{M}$ since x_0 was chosen arbitrarily.
($\Longleftarrow$) If (2.5) holds, then $(\mathcal{M}, \boldsymbol{g})$ is locally flat and thus admits a system of local coordinates for which the Levi-Civita connection coefficients satisfy $\Gamma^{i}{}_{jk} = 0$. Given a covector $\omega_i \in \mathcal{C}(\mathcal{M})$, it follows from the definition of the covariant

derivative that $\nabla_j \omega_i = \omega_{i,j}$. Thus, Eq. (2.1) reduces to $\omega_{i,j} = 0$ which has the general solution $\omega_i = c_i$, $i = 1, \ldots, n$, where the c_i are constant. We may write this general solution in the form $\omega_i = c_\alpha \omega_i^\alpha$ where $\omega_i^\alpha = \delta_i^\alpha$. Clearly, the set of covectors $\{\omega_i^\alpha\}_{\alpha=1}^n$ is covariantly constant and linearly independent. Therefore, $\dim \mathcal{C}(\mathcal{M}) = n$. □

Proposition 2. *Let* $\omega_i^1, \omega_j^2 \in \mathcal{C}(\mathcal{M})$ *and define the scalar*

$$P^{\alpha\beta} = g^{ij} \omega_i^\alpha \omega_j^\beta, \tag{2.8}$$

where $\alpha, \beta = 1, 2$. *Then,* $P^{\alpha\beta}$ *is constant.*

Proof. Immediate upon direct computation of ${P^{\alpha\beta}}_{,k}$. □

Assume now that $(\mathcal{M}, \boldsymbol{g})$ is locally flat which by Proposition 1 implies that $\dim \mathcal{C}(\mathcal{M}) = n$. The inverse pseudo-Riemannian metric $\boldsymbol{g}^{-1}$ clearly induces an inner product on $\mathcal{C}(\mathcal{M})$. Let $\{\boldsymbol{\omega}^\alpha\}_{\alpha=1}^n$ be a basis of $\mathcal{C}(\mathcal{M})$. Without loss of generality, we may assume that none of the covectors $\boldsymbol{\omega}^\alpha$ is null. By the Gram–Schmidt process and Proposition 2, we may assume that the basis is quasi-orthonormal, i.e.

$$\boldsymbol{g}^{-1}(\boldsymbol{\omega}^\alpha, \boldsymbol{\omega}^\beta) = \eta^{\alpha\beta}, \tag{2.9}$$

where

$$\eta^{\alpha\beta} = \mathrm{diag}(-1, \ldots, -1, 1, \ldots 1) \tag{2.10}$$

and the number of minus signs in (2.10) is equal to the signature of $\boldsymbol{g}$. Such a set of covectors defines a moving frame on $\mathcal{M}$ (or on an open set of $\mathcal{M}$). We may write

$$\boldsymbol{\omega}^\alpha = \omega_i^\alpha \, \mathrm{d}u^i, \tag{2.11}$$

in terms of a local system of coordinates u^i on $\mathcal{M}$. By definition, $\nabla_j \omega_i^\alpha = 0$, since the $\boldsymbol{\omega}^\alpha$ are assumed to be CC. It thus follows that $\nabla_{[j} \omega_{i]}^\alpha = \partial_{[j} \omega_{i]}^\alpha = 0$, that is

$$\mathrm{d}\boldsymbol{\omega}^\alpha = 0. \tag{2.12}$$

Thus, the covectors $\boldsymbol{\omega}^\alpha$ are closed. By the converse of Poincaré's lemma, the covectors $\boldsymbol{\omega}^\alpha$ are locally exact, which implies the existence of n functions x^α, defined at least locally on $\mathcal{M}$, such that

$$\boldsymbol{\omega}^\alpha = \mathrm{d}x^\alpha. \tag{2.13}$$

It follows from (2.11) that ${x^\alpha}_{,i} = \omega_i^\alpha$ and that $\det({x^\alpha}_{,i}) \neq 0$. By (2.9), the metric on $\mathcal{M}$ may be expressed as

$$\mathrm{d}s^2 = \eta_{\alpha\beta} \, \boldsymbol{\omega}^\alpha \odot \boldsymbol{\omega}^\beta. \tag{2.14}$$

Thus, by (2.13), we have

$$\mathrm{d}s^2 = \eta_{\alpha\beta} \, \mathrm{d}x^\alpha \, \mathrm{d}x^\beta, \tag{2.15}$$

which by (2.10) implies that the x^α define a set of pseudo-Cartesian coordinates on $(\mathcal{M}, \boldsymbol{g})$.

The theoretical basis is now in place for the determination of the transformation to a system of pseudo-Cartesian coordinates x^α from a given system of coordinates u^i on a locally flat space $(\mathcal{M}, \boldsymbol{g})$. We summarize the procedure below and present two examples in the next section.

(1) For the given components g_{ij} of the pseudo-Riemannian (flat) metric with respect to the coordinate system u^i, compute the Christoffel symbols of the second kind.
(2) Use the Christoffel symbols obtained in Step 1 to solve and obtain the general solution of (2.1) for the components of the general CC covector field. This step amounts to solving a *linear* system of PDEs.
(3) From the general solution obtained in Step 2 which involves n arbitrary constants, extract a set of n linearly independent CC covector fields $\{\hat{\boldsymbol{\omega}}^\alpha\}_{\alpha=1}^n$.
(4) Use the Gram–Schmidt process to construct from the set $\{\hat{\boldsymbol{\omega}}^\alpha\}_{\alpha=1}^n$ a new set $\{\boldsymbol{\omega}^\alpha\}_{\alpha=1}^n$ which is a quasi-orthonormal basis satisfying (2.9).

(5) Solve the equations $\mathrm{d}x^\alpha = \boldsymbol{\omega}^\alpha$ using the results from Step 4 to obtain n "potential functions" x^α, which define the transformation from pseudo-Cartesian coordinates x^α to the given coordinate system u^i.

3. Applications

We now illustrate the procedure developed in the previous section by deriving a system of pseudo-Cartesian coordinates x^α from a given (flat) metric with respect to local coordinates u^i for a case on the Euclidean plane $\mathbb{E}^2$ and a case in three-dimensional Minkowski space $\mathbb{E}^{2,1}$. Both coordinate systems have the property that the respective HJ equations for the geodesics are solvable by separation of variables.

Example 1. Consider the flat metric in $\mathbb{E}^2$ with respect to the coordinates $u^i = (u, v)$ given by

$$\mathrm{d}s^2 = a^2(\cosh^2 u - \cos^2 v)(\mathrm{d}u^2 + \mathrm{d}v^2), \tag{3.1}$$

where $0 < u < \infty$, $0 < v < 2\pi$ and $a > 0$ is a constant. The Christoffel symbols of the metric (3.1) are

$$\begin{aligned} \Gamma^u{}_{uu} = -\Gamma^u{}_{vv} = \Gamma^v{}_{uv} &= \frac{\cosh u \sinh u}{\cosh^2 u - \cos^2 v}, \\ \Gamma^v{}_{vv} = -\Gamma^v{}_{uu} = \Gamma^u{}_{uv} &= \frac{\cos v \sin v}{\cosh^2 u - \cos^2 v}. \end{aligned} \tag{3.2}$$

Expanding Eq. (2.1) using (3.2) yields

$$\begin{aligned} (\cosh^2 u - \cos^2 v)\omega_{u,u} &= -(\cosh^2 u - \cos^2 v)\omega_{v,v} = \omega_u \cosh u \sinh u - \omega_v \cos v \sin v, \\ (\cosh^2 u - \cos^2 v)\omega_{u,v} &= -(\cosh^2 u - \cos^2 v)\omega_{v,u} = \omega_v \cosh u \sinh u + \omega_u \cos v \sin v. \end{aligned} \tag{3.3}$$

Using the method of undetermined coefficients, the general solution of (3.3) is found to be

$$\boldsymbol{\omega}|_{c_1,c_2} = \omega_u \,\mathrm{d}u + \omega_v \,\mathrm{d}v, \tag{3.4}$$

where

$$\begin{aligned} \omega_u &= c_1 \sinh u \cos v + c_2 \cosh u \sin v, \\ \omega_v &= -c_1 \cosh u \sin v + c_2 \sinh u \cos v \end{aligned} \tag{3.5}$$

and c_1 and c_2 are arbitrary constants of integration. As expected from Proposition 2, the following scalar is constant:

$$P = \boldsymbol{g}^{-1}\left(\boldsymbol{\omega}|_{c_1,c_2}, \boldsymbol{\omega}|_{k_1,k_2}\right) = a^{-2}(c_1 k_1 + c_2 k_2). \tag{3.6}$$

As $\eta_{ij} = \mathrm{diag}(1, 1)$ in $\mathbb{E}^2$, it is immediate from (3.6) that an orthonormal basis of $\mathcal{C}(\mathbb{E}^2)$ with respect to the metric (3.1) is $\{\boldsymbol{\omega}^1, \boldsymbol{\omega}^2\}$ where

$$\begin{aligned} \boldsymbol{\omega}^1 &= \boldsymbol{\omega}|_{a,0} = a \sinh u \cos v \,\mathrm{d}u - a \cosh u \sin v \,\mathrm{d}v, \\ \boldsymbol{\omega}^2 &= \boldsymbol{\omega}|_{0,a} = a \cosh u \sin v \,\mathrm{d}u + a \sinh u \cos v \,\mathrm{d}v. \end{aligned} \tag{3.7}$$

To complete the calculation, we compute the potential functions x and y satisfying $\mathrm{d}x = \boldsymbol{\omega}^1$ and $\mathrm{d}y = \boldsymbol{\omega}^2$, respectively. A straightforward integration yields

$$x = a \cosh u \cos v, \qquad y = a \sinh u \sin v, \tag{3.8}$$

modulo an additive constant. Eq. (3.8) gives the familiar transformation from Cartesian coordinates to elliptic–hyperbolic coordinates defined on the Euclidean plane.

Example 2. Consider the flat metric in three-dimensional Minkowski space $\mathbb{E}^{2,1}$ with respect to the coordinates $u^i = (u, v, w)$ given by

$$\mathrm{d}s^2 = -\frac{(u^2 - v^2)(w^2 - u^2)}{u^2}\,\mathrm{d}u^2 + \frac{(u^2 - v^2)(w^2 - v^2)}{v^2}\,\mathrm{d}v^2 + \frac{(w^2 - u^2)(w^2 - v^2)}{w^2}\,\mathrm{d}w^2, \tag{3.9}$$

for $0 < v^2 < u^2 < w^2$. The non-vanishing Christoffel symbols of (3.9) are

$$\begin{aligned} &\Gamma^u{}_{ww} = -\frac{u^3(w^2-v^2)}{w^2(u^2-v^2)(w^2-u^2)}, \quad \Gamma^u{}_{vv} = \frac{u^3(w^2-v^2)}{v^2(u^2-v^2)(w^2-u^2)}, \\ &\Gamma^u{}_{uu} = \frac{w^2v^2-u^4}{u(u^2-v^2)(w^2-u^2)}, \quad \Gamma^u{}_{uv} = -\frac{v}{u^2-v^2}, \quad \Gamma^u{}_{wu} = \frac{w}{w^2-u^2} \end{aligned} \tag{3.10}$$

with $\Gamma^v{}_{jk}$ and $\Gamma^w{}_{jk}$ obtained from (3.10) by cycling over (u, v, w). Using these Christoffel symbols, we find that the general solution of (2.1) is

$$\boldsymbol{\omega}|_{c_0,c_1,c_2} = \omega_u\,\mathrm{d}u + \omega_v\,\mathrm{d}v + \omega_w\,\mathrm{d}w, \tag{3.11}$$

where

$$\begin{aligned} \omega_u &= c_0 u + c_1 vw + c_2\left(\frac{v}{w} + \frac{w}{v} - \frac{vw}{u^2}\right), \\ \omega_v &= c_0 v + c_1 wu + c_2\left(\frac{w}{u} + \frac{u}{w} - \frac{wu}{v^2}\right), \\ \omega_w &= c_0 w + c_1 uv + c_2\left(\frac{u}{v} + \frac{v}{u} - \frac{uv}{w^2}\right) \end{aligned} \tag{3.12}$$

and c_0, c_1 and c_2 are arbitrary constants. Computing the scalar

$$P = \boldsymbol{g}^{-1}\left(\boldsymbol{\omega}|_{c_0,c_1,c_2}, \boldsymbol{\omega}|_{k_0,k_1,k_2}\right) = c_0k_0 - 2(c_1k_2 + c_2k_1), \tag{3.13}$$

and using the Gram–Schmidt procedure in conjunction with (3.11)–(3.13), we find that

$$\boldsymbol{\omega}^0 = \boldsymbol{\omega}|_{0,\frac{1}{2},\frac{1}{2}}, \quad \boldsymbol{\omega}^1 = \boldsymbol{\omega}|_{0,-\frac{1}{2},\frac{1}{2}}, \quad \boldsymbol{\omega}^2 = \boldsymbol{\omega}|_{1,0,0}, \tag{3.14}$$

defines a quasi-orthonormal basis of $\mathcal{C}(\mathbb{E}^{2,1})$ with respect to the metric (3.9), such that $\boldsymbol{g}^{-1}(\boldsymbol{\omega}^\alpha, \boldsymbol{\omega}^\beta) = \eta^{\alpha\beta}$, where $\eta^{\alpha\beta} = \mathrm{diag}(-1, 1, 1)$. Finally, solving the equations $\mathrm{d}x^\alpha = \boldsymbol{\omega}^\alpha$ for the potential functions $x^\alpha = (t, x, y)$ using (3.14) yields the desired coordinate transformation

$$\begin{aligned} t &= \frac{1}{2}\left(\frac{uv}{w} + \frac{vw}{u} + \frac{wu}{v}\right) + \frac{1}{2}uvw, \\ x &= \frac{1}{2}\left(\frac{uv}{w} + \frac{vw}{u} + \frac{wu}{v}\right) - \frac{1}{2}uvw, \\ y &= \frac{1}{2}(u^2 + v^2 + w^2), \end{aligned} \tag{3.15}$$

defined up to additive constants.

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